International Journal of Applied Science and Mathematical Theory E- ISSN 2489-009X
P-ISSN 2695-1908, Vol. 10 No. 5 2024 www.iiardjournals.org Online Version

A Modified Binomial Distribution Approximation By Poisson Distribution

[1]Samson O. [Egege*2] Emmanuel Inyang and [3]Bright O. Osu
[1]Science Education department Diamond College of Education Aba Abia Nigeria
[2] A research follow department of Mathematics Abia State University,Uturu Nigeria
[3]Mathematics department Abia State University,Uturu Nigeria
sammyhandsome490@yahoo.com
DOI: 10.56201/ijasmt.v10.no5.2024.pg41.45

Abstract
A modified Binomial distribution is obtained from a generalized Binomial distribution for a special case $\alpha < 0$. In view of the approximation of simple Binomial distribution by Poisson distribution and the modified Binomial distribution approximation of this study by Poisson distribution ,it is found to be more accurate than the simple Binomial approximation by Poisson distribution provided that $A + B$ is large.
***Keywords**: A generalized Binomial,Poisson distribution, and binomial distribution approximation*

1.0 Introduction

A generalized Binomial distribution used in the study was presented by Dwass 1979 [3] . It is a discrete distribution that covers Binomial, hyper geometric and Polya distribution. This distribution depends on four parameters A, B, n and α where A and B are positive, $n,$ is a positive integer,α is an arbitrary real number satisfying $(n-1)\alpha \le A + B$ and $A^{(i)}$ and $B^{(i)}$ are not negative for $i = ,0,1 \dots . n$. where $x^{(i)} - x(x-\alpha) \dots (x-(i-1)\alpha)$. The generalized Binomial random variable is defined by

$$P_{(k)} = \frac{A^{(k)}B^{(n-k)}}{(A+B)^n} k, = 0,1 \dots . n \qquad 1.0$$

With mean and variance $\mu = \frac{nA}{A+B}$ and $\sigma^2 = \frac{nAB(A+B-n\alpha)}{(A+B)^2(A+B-\alpha)}$respectively .

Dwass [3] gave out three special cases of the distribution (1.0) for α as

i. If $\alpha = 0$ (1.0) reduced to the Binomial distribution with parameters, A ,B and n where A , B and n where AB and n are positive integers.

$$P(k) = \binom{n}{k}\left(\frac{A}{A+B}\right)^x \left(\frac{A}{A+B}\right)^{n-x} k = 0,1 \dots n \qquad 1.2$$

ii. If $\alpha > 0$ (1.0) reduced to hypergeometric distribution with parameters $A/_\alpha$,$B/_\alpha$ and n where $A/_\alpha$, $B/_\alpha$ and n are integers

$$P(k) = \frac{\binom{A/\alpha}{k}\binom{B/\alpha}{n-k}}{\binom{A/\alpha + B/\alpha}{n}}, k = 0,1 \dots min\left\{n, \frac{A}{\alpha}\right\} \qquad 1.3$$

iii. If $\alpha < 0$, (1.0) reduced to Polya distribution with parameters A, B, n and α

$$P(k) = \frac{\binom{\frac{-A}{\alpha}+k-1}{k}\binom{\frac{-B}{\alpha}+n-k-1}{n-k}}{\binom{\frac{-A}{\alpha}-\frac{-B}{\alpha}+n-1}{n}} k = 0,1 \dots n \qquad 1.4$$

It is known that the three special cases can be approximated by Poisson distribution under a certain conditions on their parameters. Thus, if the conditions on the parameters of a generalized Binomial distribution are satisfied, then the generalized Binomial can be used to obtain a modified Binomial distribution.

Poisson limit theorem states that if $n \to \infty, p \to 0$ and $\mu = np$ remains constant, then Binomial distribution converges to Poisson distribution. This means that the Binomial distribution can approximate Poisson distribution for $\mu = np$ and with parameters n and p.

This paper focus on modifying the existing approximation of Binomial distribution by Poisson distribution, by giving a modified Binomial distribution from generalized Binomial distribution. The result of this work is used to approximate Poisson distribution for $\mu = n\frac{A}{A+B}$ with parameters$n$ and $\frac{A}{A+B}$ in comparison with the existing approximation. The level of accuracy of the result is measured in the form $\left|\hat{B}_{n,\frac{A}{A+B}}(k) - P_\lambda(k)\right|$ for $k = 0,1 \dots n$, where $P_\lambda(k) = \frac{e^{-k}\lambda^k}{k!}$, $\hat{B}_{n,\frac{A}{A+B}}$is the modified distribution .

Samson et al [1][4]used generalized Binomial function (1.0) to generate a model for evaluating option price in comparison with Binomial model and a Negative hypergeometric distribution respectively

Similar content of the study can be found in [5][6]

2.0 Method

The distribution of the generalized Binomial random variable is defined as of the form (1.0) with mean and variance $\mu = \frac{nA}{A+B}$ and $\sigma^2 = \frac{nAB(A+B-n\alpha)}{(A+B)^2(A+B-\alpha)}$ respectively .

Let X be the generalized Binomial random variable .Then following [8], with probability distribution defined as

$$P_{(k)} = \frac{A^{(k)}B^{(n-k)}}{(A+B)^n} \; k, = 0,1 \dots . n \qquad 2.0$$

$$= \binom{n}{x}\frac{[A(A-\alpha)\dots(A-(k-1)\alpha)][B(B-\alpha)\dots(B-(n-k-1)\alpha)]}{(A+B)((A+B-\alpha)\dots(A+B-(n-1)\alpha)} \qquad 2.1$$

Following the method in [2] the following lemma holds

2.1 Lemma: For $k, (A+B) \in \mathrm{N} \cup \{0\}$and $0 < \frac{A}{A+B} < 1$ then the following holds

i. $\prod_{i=0}^{k-1}\left(\frac{A}{A+B}+\frac{i}{A+B}\right) = \left(\frac{A}{A+B}\right)^k\left[1+\frac{k(k-1)}{2A}\right] + O\frac{1}{(A+B)^2}$ 2.2

ii. $\prod_{i=0}^{n-k-1}\left(\frac{B}{A+B}+\frac{i}{A+B}\right) = \left(\frac{B}{A+B}\right)^{n-k}\left[1+\frac{n-k(n-k-1)}{2B}\right] + O\frac{1}{(A+B)^2}$ 2.3

iii. $\prod_{i=0}^{n-1}\left(1+\frac{i}{A+B}\right) = \left[1+\frac{n(n-1)}{2(A+B)}\right] + O\frac{1}{(A+B)^2}$ 2.4

3.0 Result

Theorem 3.0: For $k \in N \cup \{0\}$,$\mu = \frac{nA}{A+B}$ and $\forall\alpha \in \{-1, 0\}$ then $P_{(k)} = \hat{B}_{n,\frac{A}{A+B}} + O\frac{1}{(A+B)^2}$

Where $P_{(k)} = \frac{A^{(k)}B^{(n-k)}}{(A+B)^n}$ and $\hat{B}_{n,\frac{A}{A+B}} = \frac{\binom{n}{x}\frac{[A-(k-1)\alpha.....A][B-(n-k-1)\alpha....B]}{[A+B-(n-1)\alpha....A+B]}}{1+\frac{n(n-1)}{2(A+B)}}\left\{1+\frac{k(k-1)}{2A}+\frac{(n-k)(n-k-1)}{2B}\right\}$

Proof

$$P_{(k)} = \frac{A^{(k)}B^{(n-k)}}{(A+B)^n} k, = 0,1 \ldots .n \qquad 3.0$$

$$= \binom{n}{k}\frac{[A-(k-1)\alpha.....A][B-(n-k-1)\alpha....B]}{[A+B-(n-1)\alpha....A+B]} \qquad 3.1$$

$$= \binom{n}{k}\frac{A+B\left[\frac{A}{A+B}-\frac{(k-1)\alpha}{A+B}.....\frac{A}{A+B}\right]A+B\left[\frac{B}{A+B}-\frac{(n-k-1)\alpha}{A+B}...\frac{B}{A+B}\right]}{A+B\left[1-\frac{(n-1)\alpha}{A+B}...1\right]}$$

$$= \binom{n}{k}\frac{(A+B)^k(A+B)^{n-k}\left[\frac{A}{A+B}-\frac{(k-1)\alpha}{A+B}\ldots..\frac{A}{A+B}\right]\left[\frac{B}{A+B}-\frac{(n-k-1)\alpha}{A+B}\ldots\frac{B}{A+B}\right]}{(A+B)^n\left[1-\frac{(n-1)\alpha}{A+B}\ldots 1\right]}$$

For $\alpha < 0$ and by Lemma 2.1

We obtained

$$= \binom{n}{k}\frac{\prod_{i=0}^{k-1}\left(\frac{A}{A+B}+\frac{i}{A+B}\right)\prod_{i=0}^{n-k-1}\left(\frac{B}{A+B}+\frac{i}{A+B}\right)}{\prod_{i=0}^{n-1}\left(1+\frac{i}{A+B}\right)}$$

$$= \binom{n}{k}\frac{\left(\frac{A}{A+B}\right)^k\left[1+\frac{k(k-1)}{2A}+O\frac{1}{(A+B)^2}\right]\left(\frac{B}{A+B}\right)^{n-k}\left[1+\frac{(n-k)(n-k-1)}{2B}\right]+O\frac{1}{(A+B)^2}}{\left[1+\frac{n(n-1)}{2(A+B)}+O\frac{1}{(A+B)^2}\right]}$$

$$= \frac{\frac{A^{(k)}B^{(n-k)}}{(A+B)^{(n)}}}{1+\frac{n(n-1)}{2(A+B)}}\left\{1+\frac{k(k-1)}{2A}+\frac{(n-k)(n-k-1)}{2B}\right\}+O\frac{1}{(A+B)^2}$$

$$= \frac{P_{(k)}}{1+\frac{k(k-1)}{2(A+B)}}\left\{1+\frac{k(k-1)}{2A}+\frac{(n-k)(n-k-1)}{2B}\right\}+O\frac{1}{(A+B)^2}$$

$$= \frac{\binom{n}{x}\frac{[A-(k-1)\alpha.....A][B-(n-k-1)\alpha....B]}{[A+B-(n-1)\alpha....A+B]}}{1+\frac{n(n-1)}{2(A+B)}}\left\{1+\frac{k(k-1)}{2A}+\frac{(n-k)(n-k-1)}{2B}\right\}+O\frac{1}{(A+B)^2}$$

$$= \hat{B}_{n,\frac{A}{A+B}} + O\frac{1}{(A+B)^2} \blacksquare$$

If $(A+B)$ is large, then $O\frac{1}{(A+B)^2} \approx 0$

4.0 Numerical Results

The following examples are used to illustrate how accurate the modified Binomial distribution approximates Poisson distribution.

Example 4.1 : Suppose $A + B = 100, A = 10, n = 10$ $\frac{A}{A+B} = 0.1$ and $\lambda = 1.0$

k	$P_\lambda(k)$	$\hat{B}_{n,\frac{A}{A+B}}$	$B_{n,\frac{A}{A+B}}$	$\left\|\hat{B}_{n,\frac{A}{A+B}}(x) - P_\lambda(k)\right\|$	$\left\|B_{n,\frac{A}{A+B}}(x) - P_\lambda(k)\right\|$
0	0.36787944	0.36070183	0.34867844	0.00717761	0.01920100

1	0.36787944	0.37406116	0.38742049	0.00618172	0.01954105
2	0.18393972	0.18851495	0.19371024	0.00457523	0.00977052
3	0.06131324	0.06069423	0.05739563	0.00061901	0.00973569
4	0.01532831	0.01359756	0.01116026	0.00173075	0.00416805
5	0.00306566	0.00216649	0.00148803	0.00089917	0.00157763
6	0.00051094	0.00024389	0.00013778	0.00026705	0.00086686
7	0.00007299	0.00001891	0.00000875	0.00005408	0.00006424
8	0.00000912	0.00000095	0.00000036	0.00000817	0.00000876
9	0.00000101	0.00000003	0.00000001	0.00000098	0.00000100
10	0.00000010	0.00000000	0.00000000	0.00000010	0.00000010

Example 4.2: Suppose $A+B=100 n=20, A=20\ \frac{A}{A+B}=0.2$ and $\lambda=4$

k	$P_\lambda(k)$	$\hat{B}_{n,\frac{A}{A+B}}$	$B_{n,\frac{A}{A+B}}$	$\left\lVert \hat{B}_{n,\frac{A}{A+B}}(x) - P_\lambda(k) \right\rVert$	$\left\lVert B_{n,\frac{A}{A+B}}(x) - P_\lambda(k) \right\rVert$
0	0.01831564	0.01739321	0.01152922	0.00092243	0.00678642
1	0.07326256	0.06236705	0.05764608	0.01089551	0.01561648
2	0.14652511	0.13986005	0.13690942	0.00666506	0.00961569
3	0.19536681	0.20182338	0.20536414	0.00645657	0.00999733
4	0.19536681	0.21806753	0.21819940	0.02270072	0.02283259
5	0.15629345	0.16929234	0.17455952	0.01299889	0.01826607
6	0.10419563	0.10387457	0.10909970	0.00032106	0.00490407
7	0.05954036	0.05486335	0.05454985	0.00467701	0.00499050
8	0.02977018	0.02394396	0.02216087	0.00582622	0.00760931

Example 4.3:Suppose $A+B=500, n=30,\ \ A=30\ \ \frac{A}{A+B}=0.06\ \ \lambda=1.8\,0$

k	$P_\lambda(k)$	$\hat{B}_{n,\frac{A}{A+B}}$	$B_{n,\frac{A}{A+B}}$	$\left\lVert \hat{B}_{n,\frac{A}{A+B}}(x) - P_\lambda(k) \right\rVert$	$\left\lVert B_{n1,\frac{A}{A+B}}(x) - P_\lambda(k) \right\rVert$
0	0.16529889	0.16258931	0.15625561	0.00270953	0.00904328
1	0.29753800	0.29786527	0.29921286	0.00032727	0.00167486
2	0.26778420	0.27037944	0.27693105	0.00259524	0.00914685
3	0.16067052	0.16218740	0.16498020	0.00151688	0.00430968
4	0.07230173	0.07219686	0.07108190	0.00010487	0.00121983
5	0.02602862	0.02534336	0.02359314	0.00068526	0.00243548
6	0.00780859	0.00726973	0.00627477	0.00053886	0.00153382
7	0.00200792	0.00174236	0.00137320	0.00526181	0.0063472
8	0.00045178	0.00035405	0.00025200	0.00009773	0.00019978
9	0.00009036	0.00006161	0.00003932	0.00002875	0.00005104
10	0.00001626	0.00000925	0.00000527	0.00000701	0.00001099

From the examples 4.1-4.3 , it is seen that the modified Binomial distribution approximate Poisson more sufficient enough more than the simple Binomial distribution

5.0 Conclusion

The result of this study is a modified Binomial distribution with parameters $n, A + B$ and $\frac{A}{A+B}$ using a generalizedBinomial distribution. In view of the result, it is observed that if $\frac{A}{A+B}$ is small or $A + B$ is large, the result gives a good approximation, so to say that the modified Binomial distribution can be used to approximate Poisson distribution provided $A + B$ is large and $\frac{A}{A+B}$ is small.

References

[1]Bright O. Osu , Samson O. Egege and Emmanuel j. Ekpeyong, " Application Of Generalized Binomial Distribution Model for Option Pricing, American Journal of Applied Mathematics and Statistics ,5(2) pp 67-71 2017

[2]D.P Hu , Y.Q Cui and H. Yin ,"An improved Negative Binomial Approximation for Negative Hypergeometric Distribution" Applied Mechanics and Material Vol 427-429, pp 2549-2553 ,2013

[3] Dwass .M , "A generalized Binomial distribution" American Statisticians Vol 33,89-87 1979

[4]Samson O.Egege ,Bright O.Osu and Carlos Granados, " Application of a limit Function of negative hypergeometric distribution in option pricing" BISTUA 20(2) ,pp 43-47 2022

[5]Samson O.Egege , Bright O. Osu ,Kingsley Uchendu and Chiemela B. Akachi, " An improved Poisson approximation for the generalized Binomial Distribution with Financial Application, Elixir International Journal Applied Mathematics , Vol 121 pp51509-51519 , 2018

[6]Samson O. Egege , Bright O.Osu and Chigozie Chibuisi, " An Improved Poisson distribution and its Application in Option pricing ,Open Science Journal of mathematics and Application 6(3) , pp15-23 2018

[7]Teerapabolarn .K., "An improved Binomial distribution to approximate the Polya distribution" , International journal of Pure and Applied Mathematics Vol 93, N0 .5 pp 629-632 ,2014

[8] Wongkasem P. and Teerapabolarn K. "On Approximating a Generalized Binomial and Poisson Distribution" International Journal of Statistics and Systems Vol 3 No. 2 pp113-124 2008